An Axiomatic and Data Driven View on the EPK Paradox

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Motivation

- Pricing kernel (PK)
  - Consumption based models
    - marginal rate of consumption substitution
  - Arbitrage free models
    - Radon-Nikodym derivative of the physical measure w.r.t. the risk neutral measure ▸ Risk Neutral Valuation ▸ PK - Black-Scholes

- Empirical pricing kernel (EPK)
  - Any estimate of the PK
  - EPK paradox - locally increasing EPK

EPK Paradox

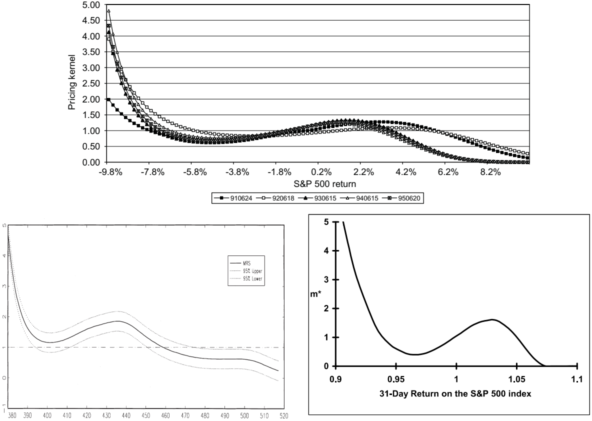

Figure 1: EPK's: Engle and Rosenberg (2002), Ait-Sahalia and Lo (2000), Brown and Jackwerth (2004)

EPK Paradox

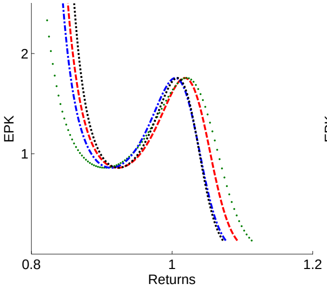

Figure 2: EPK's for various maturities (left) and different estimation dates for fixed maturity 1M (right), Grith et al. (2010)

EPK Paradox

Figure 3: EPK's across moneyness κ and maturity τ for DAX from 20010101 – 20011231, Giacomini and Härdle (2008)

EPK Paradox

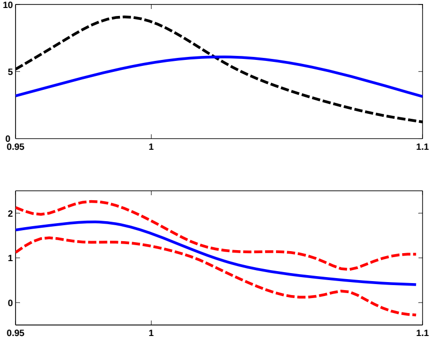

Figure 4: Upper panel: estimated risk neutral density $\widehat{q}$ and historical density $\widehat{p}$. Lower panel: EPK and 95% uniform confidence bands on 20080228, Härdle et al. (2010)

Objectives

- Pricing kernel derivation
  - Adjust individual and aggregate preferences
  - State-dependent (state variable: market return)
  - Simulation study
- Fitting EPK's
  - Test function approach
  - Empirical study

Research Questions

- How to modify standard expected utility theory to rationalize the EPK paradox?
- How well can 'observed' EPK's be fitted?
- How sensitive are results with respect to the preference parameters?

Outline

1. Motivation ✓
2. Microeconomic Framework
3. Pricing Kernel
4. Fitting EPK's
5. Empirical Study
6. Statistical Properties
7. Conclusions

Assumptions

- Financial markets
  - Finite investment time horizon $[0, T]$ and r risk free interest rate
  - Risky asset with prices $\{S_t\}_{0 \leq t \leq T}$ and return $R_T = S_T / S_0$
  - Arbitrage free market, at least one equivalent martingale measure with density π
- m Consumers
  - Endowment $e_i(R_T)$ and consumption $c_i(R_T)$, $i = 1, \ldots, m$
  - State-dependent utility function

State-Dependent Utility - Literature Review

- Axiomatisation
  - Dreze and Rustichini (2004)
  - Evans and Viscusi (1991)
  - Mas-Colell, Winston und Green (1995)
- Empirical evidence
  - Karni, Schmeidler and Vind (1983)

Individual Preferences

- Consumer i's extended expected utility, Mas-Colell et al. (1995)

$$U^i \left\{ c_i \left(R_T \right) \right\} = \mathsf{E} \left[u^i \left\{ R_T, c_i \left(R_T \right) \right\} \right],$$

with $u^i : \mathbb{R}^2_+ \rightarrow \mathbb{R}$ - state dependent utility index

$$u^i \left\{ R_T, c_i \left(R_T \right) \right\} = u^0_i \left\{ c_i \left(R_T \right) \right\} \mathbf{I} \left\{ R_T \in [0, x_i] \right\} + u^1_i \left\{ c_i \left(R_T \right) \right\} \mathbf{I} \left\{ R_T \in (x_i, \infty) \right\}$$

$x_i \in [0, \infty)$ - reference point of consumer i; $x_1 \leq \cdots \leq x_m$
u^0_i, u^1_i : $\mathbb{R}_+ \rightarrow \mathbb{R}$ - utility indices
- strictly increasing, concave and twice cts differentiable

Individual Preferences

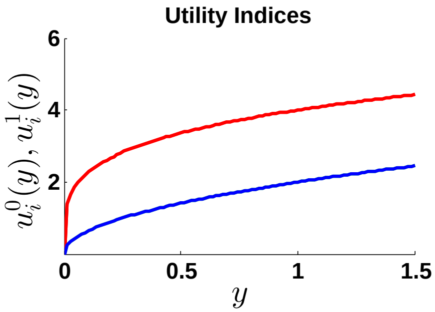

Figure 5: Utility indices $u_i^0(y) = y^{0.25}/0.25$ (bearish market) and $u_i^0(y) = y^{0.50}/0.50$ (bullish market)

Equilibrium

- Individual optimization

$$\bar{c}_i(R_T) = \arg\max_{c_i(R_T)} U^i\{c_i(R_T)\}$$

- Market clearing

$$\sum_{i=1}^{m} \bar{c}_i(R_T) = \sum_{i=1}^{m} e_i(R_T) \stackrel{\text{def}}{=} \bar{e}(R_T)$$

  - Pareto optimal $\bar{c}_1(R_T), \ldots, \bar{c}_m(R_T)$

Aggregated Preferences

- Aggregated extended expected preferences

$$U_\alpha \{\bar{e}(R_T)\} = \mathsf{E}\left[u_\alpha \{R_T, \bar{e}(R_T)\}\right],$$

with $u_\alpha : \mathbb{R}^2_+ \to \mathbb{R}$ - aggregated indirect utility

$$\begin{aligned} u_\alpha \{R_T, \bar{e}(R_T)\} = & u_{\alpha,1} \{\bar{e}(R_T)\} \mathbf{I}\{R_T \in [0, x_1]\} + \\ & + \sum_{i=1}^{m-1} u_{\alpha,i+1} \{\bar{e}(R_T)\} \mathbf{I}\{R_T \in (x_i, x_{i+1}]\} + \\ & + u_{\alpha,m+1} \{\bar{e}(R_T)\} \mathbf{I}\{R_T \in (x_m, \infty)\} \end{aligned}$$

$$u_{\alpha,j} \{\bar{e}(R_T)\} = \sum_{k=1}^{m} \alpha_k u_k^0 \{\bar{c}_k(R_T)\} \mathbf{I}\{k \geq j\} + \sum_{k=1}^{m} \alpha_k u_k^1 \{\bar{c}_k(R_T)\} \mathbf{I}\{k < j\}$$

for $j = 1, \ldots, m+1$ and importance weights $\alpha = (\alpha_1, \ldots, \alpha_m)^\top$

Pricing Kernel

Theorem

If $\bar{e}\left(r_T\right) = r_T$ then for every realization r_T of R_T

$$
\begin{aligned}
\widetilde{\mathcal{K}}_\pi\left(r_T\right) = & \left.\frac{\partial u_{\alpha,1}\left\{y\right\}}{\partial y}\right|_{y=r_T} \mathbf{I}\left\{r_T \in [0, x_1]\right\} + \\
& + \sum_{i=1}^{m-1} \left.\frac{\partial u_{\alpha,i+1}\left\{y\right\}}{\partial y}\right|_{y=r_T} \mathbf{I}\left\{r_T \in \left(x_i, x_{i+1}\right]\right\} + \\
& + \left.\frac{\partial u_{\alpha,m+1}\left\{y\right\}}{\partial y}\right|_{y=r_T} \mathbf{I}\left\{r_T \in \left(x_m, \infty\right)\right\}.
\end{aligned}
$$

Note: $\widetilde{\mathcal{K}}_\pi\left(r_T\right)$ is nonincreasing separately on the intervals $\left[0, x_1\right], \left(x_1, x_2\right], \ldots, \left(x_m, \infty\right)$ but may be nonmonotone at x_i's

Risk Aversion

- Arrow-Pratt coefficients of absolute risk aversion (ARA) at x_i's from the left and from the right

$$\lim_{\delta \to 0_+} \frac{\frac{\widetilde{\mathcal{K}}_\pi(x_i-\delta)-\widetilde{\mathcal{K}}_\pi(x_i)}{\delta}}{\widetilde{\mathcal{K}}_\pi(x_i)}, \quad \lim_{\delta \to 0_+} \frac{\frac{\widetilde{\mathcal{K}}_\pi(x_i+\delta)-\widetilde{\mathcal{K}}_\pi(x_i)}{\delta}}{\widetilde{\mathcal{K}}_\pi(x_i)}$$

Example 1

Example 1. Consider m investors with identical reference point x_1 that switch between constant relative risk aversion (CRRA) utilities $u_i^0(y) = y^{\gamma_i^0}/\gamma_i^0$ and $u_i^1(y) = y^{\gamma_i^1}/\gamma_i^1$, $0 < \gamma_i^0 < \gamma_i^1 < 1$.

$$\widetilde{\mathcal{K}}_\pi(r_T) = r_T^{\gamma_\alpha^0 - 1}\,\mathbf{I}\{r_T \in [0, x_1]\} + r_T^{\gamma_\alpha^1 - 1}\,\mathbf{I}\{r_T \in (x_1, \infty)\},$$

$\gamma_\alpha^\ell - 1 = r_T / \sum_{i=1}^{m} \frac{\bar{c}_i(r_T)}{\gamma_i^\ell - 1}$, $\ell = \{0, 1\}$ - implied CRRA coeff's

Example 1

▸ R code

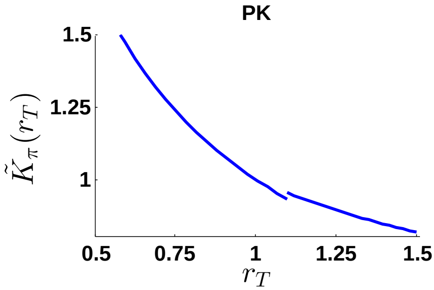

Figure 6: Pricing kernel $\widetilde{\mathcal{K}}_\pi\left(r_T\right)$ for $x_1 = 1.1$ and $\gamma_\alpha^0 = 0.25 < \gamma_\alpha^1 = 0.50$

Example 2

Example 2. Consider m investors with possibly different reference points x_i's that switch between CRRA utilities $u^0(y) = b_0 \frac{y^\gamma}{\gamma}$ and $u^1(y) = b_1 \frac{y^\gamma}{\gamma}$. Let $F(r_T)$ be the cdf of the reference points

$$F(r_T) = m^{-1} \sum_{i=1}^{m} \mathbf{I}\{x_i \leq r_T\}$$

$$\mathcal{K}_{v,F}(r_T) = \widetilde{\mathcal{K}}_\pi(r_T) = \left[\frac{r_T}{\{1 - F(r_T)\} b_0^{\frac{1}{1-\gamma}} + F(r_T) b_1^{\frac{1}{1-\gamma}}} \right]^{\gamma - 1} \quad (1)$$

for parameters $v = (\gamma, b_0, b_1)^\top$, $0 < b_0 \leq b_1$

Example 2 ▸ R code

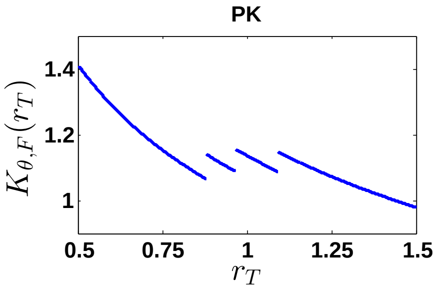

Figure 7: Pricing kernel $\mathcal{K}_{v,F}(r_T)$ with $\gamma = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 3$ with uniformly generated reference points

Example 2 ▸ R code

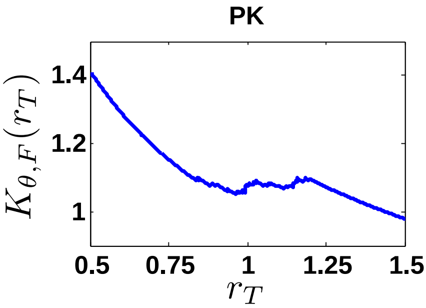

Figure 8: Pricing kernel $\mathcal{K}_{v,F}(r_T)$ with $\gamma = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 40$ with reference points generated from a triangular distribution

Example 2 R code

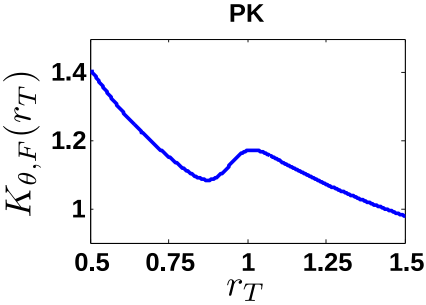

Figure 9: Pricing kernel $\mathcal{K}_{v,F}(r_T)$ with $\gamma = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 40$ with reference points generated from a normal distribution

Example 2

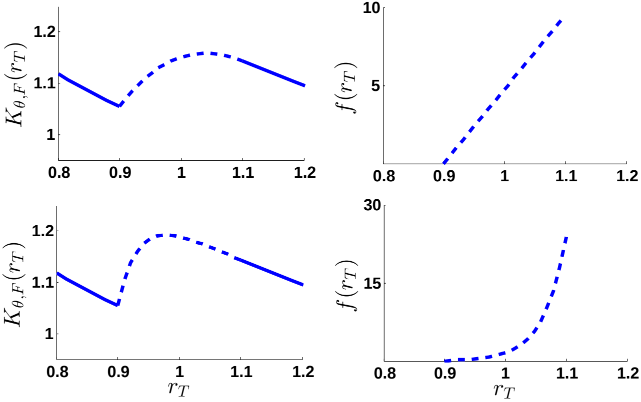

Figure 10: $\mathcal{K}_{v,F}(r_T)$ and rescaled pdf's $f(r_T) = \partial F(r_T)/\partial r_T$ for the linear and exponential specifications with $\gamma = 0.5$, $b_0 = 1$ and $b_1 = 1.2$

Example 2

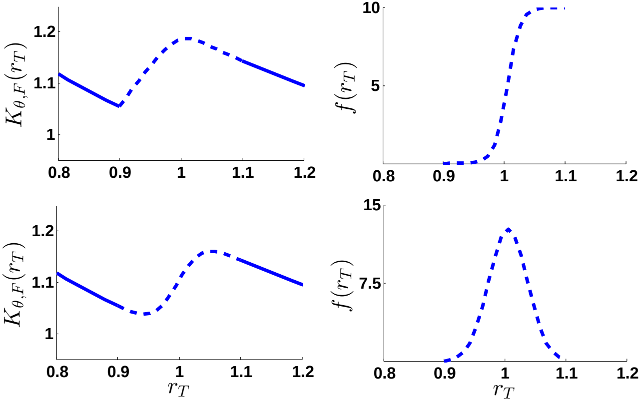

Figure 11: $\mathcal{K}_{v,F}(r_T)$ and rescaled pdf's $f(r_T) = \partial F(r_T)/\partial r_T$ for the logistic and bell shaped specifications with $\gamma = 0.5$, $b_0 = 1$ and $b_1 = 1.2$

Fitting EPK's

Test function approach, e.g., Izuchi (1997)

- Family $\mathcal{V}$ of strictly decreasing C^1-mappings $v : [0, \infty) \to \mathbb{R}$ with $\lim_{x \to 0} v(x) = \infty$ and $\lim_{x \to \infty} v(x) = 0$
- $\mathcal{Z}_N$ - set of partitions $\boldsymbol{x}_N = (x_1, \ldots, x_N)^\top$ with $N \leq m$ switching points , $\underline{z} = x_0 < x_1 < \cdots < x_N < x_{N+1} = \overline{z}$
- Test function for $\widetilde{\mathcal{K}}_\pi(x)$ approximation to fit an observed $\widehat{\mathcal{K}}(x)$

$$\sum_{i=1}^{N+1} v_i(x) \mathbf{I}\{x \in (x_{i-1}, x_i]\}$$

Fitting EPK's

- Find $\boldsymbol{x}_N^*, v_1^*, \ldots, v_{N+1}^*$ that minimize

$$\int \left\{ \widehat{\mathcal{K}}(x) - \sum_{i=1}^{N+1} v_i(x) \mathbf{I}\{x \in (x_{i-1}, x_i]\} \right\}^2 p(x)\, dx$$

for observed $\widehat{\mathcal{K}}(x)$ and p - pdf of X

- Grid-based approach - find $\widehat{v}$ and $\widehat{F}$ that minimize

$$\sum_{j=1}^{n} \left\{ \widehat{\mathcal{K}}(s_j) - \mathcal{K}_{v,F}(s_j) \right\}^2 \widehat{p}(s_j)(s_j - s_{j-1})$$

for observation points $\{s_j\}_{j=0}^n$, see Example 2 (**??**)

Data

- Financial markets
  - EUREX European option data on 20000920 and 20060621
  - Daily DAX returns - past 500 observations until 20000920 and 20060621 respectively
- Pricing kernels
  - $\widehat{\mathcal{K}}(r_T)$ - Grith et al. (2010)
  - $\mathcal{K}_{v,F}(r_T) = \widetilde{\mathcal{K}}_\pi(r_T)$ - semi-parametric PK (**??**)
  - $\mathcal{K}_{\widehat{v},\widehat{F}}(r_T)$ - estimated $\mathcal{K}_{v,F}(r_T)$

Fitting Results

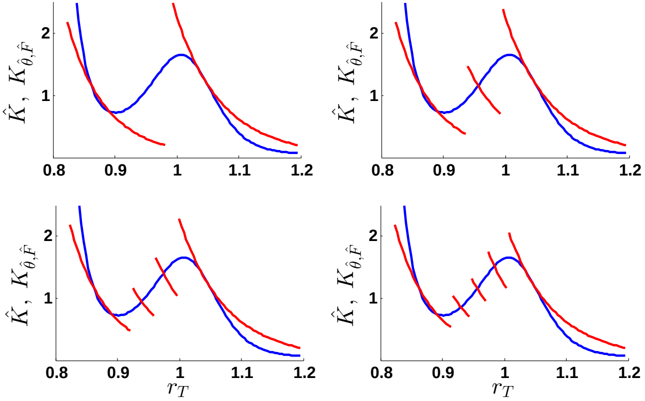

Figure 12: $\widehat{K}\,(r_T)$ on 20000920 and $\mathcal{K}_{\widehat{v},\widehat{F}}\,(r_T)$ for $m = 1, 2, 3, 4$.
$\widehat{v} = (-12.61, 0.15, 2.27)^{\top}$

Fitting Results

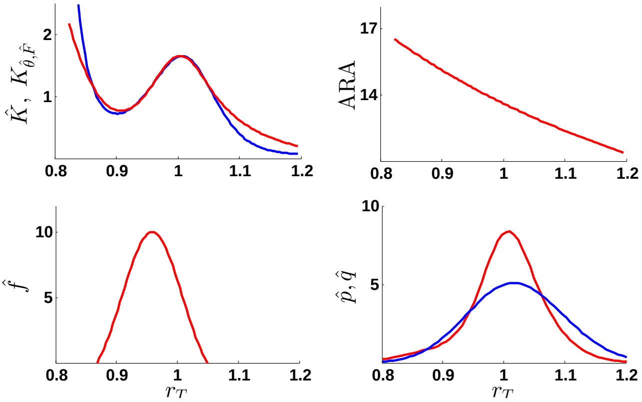

Figure 13: $\widehat{K}(r_T)$ on 20000920 and $\mathcal{K}_{\widehat{v},\widehat{F}}(r_T)$ with implied ARA. Rescaled $\widehat{f}$ and $\widehat{p}$, $\widehat{q}$. $\widehat{v} = (-12.61, 0.15, 2.27)^\top$

Fitting Results

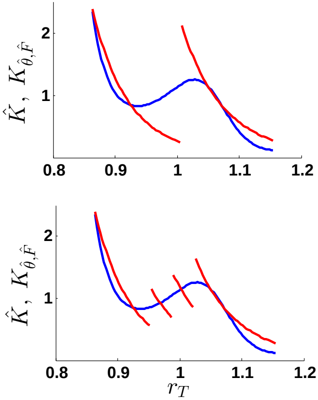

Figure 14: $\widehat{K}(r_T)$ on 20060621 and $\mathcal{K}_{\widehat{v},\widehat{F}}(r_T)$ for $m = 1, 2, 3, 4$.
$\widehat{v} = (-13.96, 0.27, 2.38)^\top$

Fitting Results

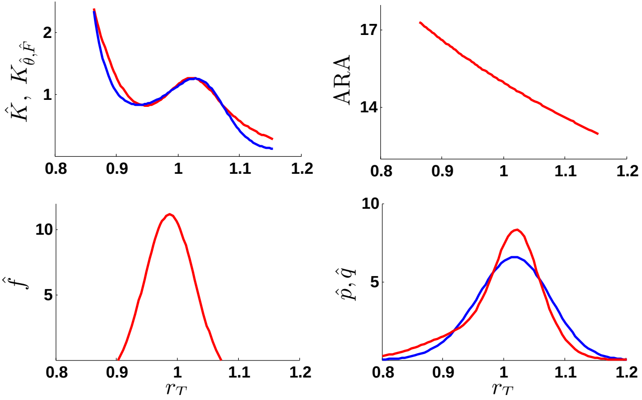

Figure 15: $\widehat{K}(r_T)$ on 20060621 and $\mathcal{K}_{\widehat{\theta},\widehat{F}}(r_T)$ with implied ARA. Rescaled $\widehat{f}$ and $\widehat{p}$, $\widehat{q}$. $\widehat{v} = (-13.96, 0.27, 2.38)^\top$

Semiparametric Pricing Kernel

$$\mathcal{K}_{v,F}(x) = \left[\frac{x}{\{1 - F(x)\} b_0^{\frac{1}{1-\gamma}} + F(x) b_1^{\frac{1}{1-\gamma}}} \right]^{\gamma - 1}$$

with $v = (\gamma, b_0, b_1)^\top$ and F cdf.

$$F(x) = \int_0^x \sum_{k=1}^p \beta_k \psi_k(u) du = \sum_{k=1}^p \beta_k \int_0^x \psi_k(u) du = \sum_{k=1}^p \beta_k \Psi_k(x)$$

with p fixed. Then

$$G(x;\theta) \stackrel{\text{def}}{=} \mathcal{K}_{v,F}(x), \quad \theta = (\gamma, b_0, b_1, \beta_1, \dots, \beta_p)^\top$$

Log-likelihood Approach

Assume that for $y_j = \widehat{\mathcal{K}}(x_j)$, $j = 1, \ldots, n$ it holds

$$y_j = G(x_j; \theta) + \varepsilon_j$$

- $G(\cdot)$ is twice differentiable
- exponential moments of ε_j exists

For $Y = (y_1, \ldots, y_n)^\top$ and $X = (x_1, \ldots, x_n)^\top$

$$L(X, Y; \theta) = L(\theta) = -\sum_{j=1}^{n} \{y_j - G(x_j, \theta)\}^2$$

$$\tilde{\theta} = \arg \max_\theta L(\theta) \quad \text{and} \quad \theta^* = \arg \max_\theta \mathrm{E}\, L(\theta)$$

Finite Sample Theory

$$L(\theta) = \underbrace{\mathsf{E}\, L(\theta)}_{\text{deterministic function}} + \underbrace{\zeta(\theta)}_{\text{stochastic component}}$$

Excess log-likehood

$$L(\theta, \theta^*) = L(\theta) - L(\theta^*)$$

helps explain statistical properties of MLE $\tilde{\theta}$ within a local set $\Theta_0(r) = \{\theta : ||V_0(\theta - \theta^*)|| \leq r\}$ for $V_0 = \mathsf{Var}\{\nabla\zeta(\theta^*)\}$.

Under some smoothness conditions for $\mathsf{E}\, L(\theta)$ and $\zeta(\theta)$

$$L(\theta, \theta^*) \approx (\theta - \theta^*)^\top \nabla\zeta(\theta^*) - ||D_0(\theta - \theta^*)||^2/2$$

for $D_0^2 = -\nabla^2\, \mathsf{E}\, L(\theta^*)$, Spokoiny (2012)

Conclusions

Pricing kernel derivation

- Reference points determine jumps in the aggregate utility
- State-dependent preferences may explain the EPK paradox

Fitting EPK's

- Quality increases with the number of switching points
- Semi-parametric PK specification successfully applied

Conclusions

Further Research

- Multidimensional reference points
- Dynamic implementation (PK's, reference points)
- Statistical estimation methodology for semi-parametric PK's
- Theoretical properties of $\widehat{v}$ and $\widehat{F}$

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References

Aït-Sahalia, Y. and Lo, A.
Nonparametric Risk Management and Implied Risk Aversion
Journal of Econometrics **94**: 9–51, 2000

Brown, D. and Jackwerth, J.
The Pricing Kernel Puzzle: Reconciling Index Option Data and Economic Theory
Discussion paper, University of Konstanz, 2004

Engle, R. and Rosenberg, J.
Empirical Pricing Kernels
Journal of Financial Economics **64**(3): 341–372, 2002

References

Dreze, J.H. and Rustichini, A.
State-Dependent Utility and Decision Theory, 2004
Handbook of Utility Theory: Extentions **2**, Springer, 2010

Evans, W.N. and Viscusi, W.K.
Estimation of State-Dependent Utility Using Survey Data
The Review of Economics and Statistics **73**(1): 94–104, 1983

Karni, E., Schmeidler, D. and Vind, K.
On State Dependent Preferences and Subjective Probabilities
Econometrica **51**(4): 1021–1031, 1983

References

Giacomini, E. and Härdle, W. K.
Dynamic Semiparametric Factor Models in Pricing Kernel Estimation
in Functional and Operational Statistics, Dabo-Niang, S. and Ferraty, F. (Eds), Contributions to Statistics, Springer Verlag ISBN 978-3-7908-2061-4. 181-187, 2008

Grith, M., Härdle, W. K. and Park, J.
Shape Invariant Modelling of Pricing Kernels and Risk Aversion
Re-submitted to the Journal of Financial Econometrics on 15.09.2011.

References

Härdle, W.K., Okhrin, Y. and Wang, W.
Uniform Confidence Bands for Pricing Kernels
Submitted to the Journal of Empirical Finance on 01.04.2010.

Izuchi, K.
Sequential Type Korovkin Theorem on $L\infty$ for QC-Test Functions
Proceedings of the American Mathematical Society **125**(4): 1153–1159, 1997

Mas-Colell, A., Whinston, M. D., and Greene, J. R.
Microeconomic Theory
Oxford University Press, 1995

Risk Neutral Valuation

▸ Motivation

- Present value of the payoffs $\psi(S_T)$

$$P_0 = \mathsf{E}_Q\left[e^{-Tr}\psi(s_T)\right] = \int_0^\infty e^{-Tr}\psi(s_T)\,\mathcal{K}(s_T)p(s_T)\,ds_T$$

r risk free interest rate, $\{S_t\}_{t\in[0,T]}$ stock price process,
p pdf of S_T, Q risk neutral measure, $\mathcal{K}(\cdot)$ pricing kernel

PK under the Black-Scholes Model

▸ Motivation

- Geometric Brownian motion for S_t

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

μ mean, σ volatility, W_t Wiener process

- Physical density p is log-normal, $\tau = T - t$

$$p_t(S_T) = \frac{1}{S_T\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{1}{2}\left\{\frac{\log(S_T/S_t) - \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right\}^2\right]$$

- Risk neutral density q is log-normal: replace μ by r

PK under the Black-Scholes Model

▸ Motivation

- PK is a decreasing function in S_T for fixed S_t

$$\mathcal{K}(S_t, S_T) = \left(\frac{S_T}{S_t}\right)^{-\frac{\mu-r}{\sigma^2}} \exp\left\{\frac{(\mu-r)\left(\mu+r-\sigma^2\right)\tau}{2\sigma^2}\right\}$$
$$= b\left(\frac{S_T}{S_t}\right)^{-\delta}$$

$b = \exp\left\{\frac{(\mu-r)\left(\mu+r-\sigma^2\right)\tau}{2\sigma^2}\right\}$ and $\delta = \frac{\mu-r}{\sigma^2} \geq 0$ constant relative risk aversion (CRRA) coefficient

Example 1

▸ Example 1

```
# Step 1/3: Input parameters
R = t(matrix(seq(0.8, 1.2, by = 0.01), 1))
x0 <- 1.1
gamma0 <- 0.25
gamma1 <- 0.50


# Step 2/3: Define the PK
K = R[R <= x0, ] ^ (gamma0 - 1)
K2 = R[R >= x0, ] ^ (gamma1 - 1)

# Step 3/3: Plot the PK against simple gross market return
plot(R[R <= x0, ], K, type = 'l', lwd = 3, col = "blue",
 xlim = c(0.8, 1.2), ylim = c(0.8, 1.25), xlab = "r_T")
lines(R[R >= x0, ], K2, type = 'l', lwd = 3, col = "blue",
 xlim = c(0.8, 1.2), ylim = c(0.8, 1.25), xlab = "r_T")
```

Example 2

Example 2

```
gamma = 0.5
b0 = 1
b1 = 1.2

# Step 1/3: Input parameters and F_n
n = 1000
s = seq(0.5, 1.5, 0.2/n)
m = 10 # number of switching points
x = runif(m, 0.8, 1.2)
F_n = ecdf(x)(s)

# Step 2/3: Define the PK
PK = (s/((1 - F_n)*b0^(1/(1-gamma)) + F_n*b1^(1/(1-gamma))))^(gamma-1)

# Step 3/3: Plot the PK against simple gross market return
plot( cbind(s, PK) )
```

Example 2

▸ Example 2

```
gamma = 0.5
b0 = 1
b1 = 1.2

# Step 1/3: Input parameters and F_n
n = 1000
s = seq(0.5, 1.5, 0.2/n)
m = 40 # number of switching points
x = 0.8 + 0.4*sqrt(runif(m))
F_n = ecdf(x)(s)

# Step 2/3: Define the PK
PK = (s/((1 - F_n)*b0^(1/(1-gamma)) + F_n*b1^(1/(1-gamma))))^(gamma-1)

# Step 3/3: Plot the PK against simple gross market return
plot( cbind(s, PK) )
```

Example 2

Example 2

```
gamma = 0.5
b0 = 1
b1 = 1.2

# Step 1/3: Input parameters and F_n
n = 1000
s = seq(0.5, 1.5, 0.2/n)
m = 40 # number of switching points
F_n = pnorm( 20*(s-0.95) )

# Step 2/3: Define the PK
PK = (s/((1 - F_n)*b0^(1/(1-gamma)) + F_n*b1^(1/(1-gamma))))^(gamma-1)

# Step 3/3: Plot the PK against simple gross market return
plot( cbind(s, PK) )
```